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Exact Stationary States of a Two-Dimensional Transport Model

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Previous calculations of transport coefficients of particle systems in two or more dimensions have used computer simulations or various approximations, like those implicit in the Boltzmann equation or in the standard probability theory of diffusion. Here an exactly solved model of crystal growth in three dimensions is transformed into a 2D model of steady particle transport. Then the exact, many-body stationary distribution and exact transport coefficients can be found as well as correlation functions. The model is highly asymmetric. In the transverse direction, particles are strongly attracted, so that long transverse rows of particles tend to form at low temperature. Kinks in these rows are rare, so the transverse positions of kinks effectively move continuously on a large distance scale.

KEY WORDS: Transport coefficient; stationary distribution; steady-state current.

1. INTRODUCTION

Simple particle transport models have many applications, including granular flow, fast ionic conduction, flow of charged droplets in microemulsions and transport in biological systems; Here we introduce a 2-dimensional particle transport model for which the exact stationary distribution can be found leading to a fairly complete description of steady state transport in the model. It appears be the first example of such a solution where the particles have a 2-dimensional interaction, other than coupled 1-dimensional diffusion processes.

The present model is partly motivated by difficulties with the much studied driven lattice gas in 2-dimensions.[1, 2] In particular, it's stationary

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distribution is unknown, i.e., there is no known equivalent of the Gibbs distribution. Moreover, one can show, by the methods of ref. 3 that the stationary distribution cannot be Gibbs-like. It is likely to be complicated, and might never be obtained in explicit form. Some mathematical progress has been made; e.g., the calculation of high-temperature perturbations relative to the ideal (non-interacting) model.[4] Low temperature analysis is more difficult, because of the strong influence of particle interactions. We shall give an argument suggesting that our new model is related to a lattice gas in the limit of extreme anisotropy in the interactions, but be we do not prove this rigorously.

The new model is mathematically equivalent to a solvable model of 3D crystal growth.[5] The purpose of this note is merely to point out this analogy and give some formulae for the current. The reader may consult results published elsewhere.

2. THE MODEL

The gas occupies a rectangular region of width L and integer height Ω, and flows in the upward direction. Configurations are defined via Λ *contours* $l = 1, 2,..., \Lambda$, as in Fig. 1a. Periodic (toroidal) boundary conditions (PBC's) apply in both directions. Thus contour $\Lambda + 1$ is identified with contour 1. The periodic image of contour Λ is shown as a broken line. Contour l, has N_l unit *downkinks* facing right at (real valued) locations $y_{l1},..., y_{l, N_l}$ and $N_l + K$ unit *upkinks* facing left at locations $x_{l1},..., x_{l, N_l + K}$. Contours do not touch; i.e., there is always at least unit vertical distance separating two contours.

The gas itself is constructed by colouring (black) a strip of unit height below each contour (Fig. 1b) representing the particles. Because of the unit gap between contours, the black regions do not overlap, but they can touch. The particles have zero width and form a continuum in the horizontal direction, as discussed below. One can see (Fig. 1) that there is a one-to-one correspondence between the black and white patchwork and the contours (of course, only those patchworks generated by contours are allowed). If $K \neq 0$, the flow is, in effect, inclined relative to the vertical.

The configuration or state $\mathbf{C}$ of the process comprises these variables, subject to $0 \leqslant x_{l1} \leqslant \cdots \leqslant x_{l, N_l + K} < L$ and $0 \leqslant y_{l1} \leqslant \cdots \leqslant y_{l, N_l} < L$ for each l, together with the vertical coordinates $\zeta_1,..., \zeta_\Lambda$ of the contours on the left hand border, subject to $1 \leqslant \zeta_1 < \cdots < \zeta_\Lambda \leqslant \Omega$ and $\zeta_\Lambda < \zeta_1 + \Omega - 1$.

Upward flow occurs via 3 types of transitions. A vertical black strip of unit height and infinitesimal width (i.e., a particle) makes unit upward displacement. There must have been a white space available for this to happen;

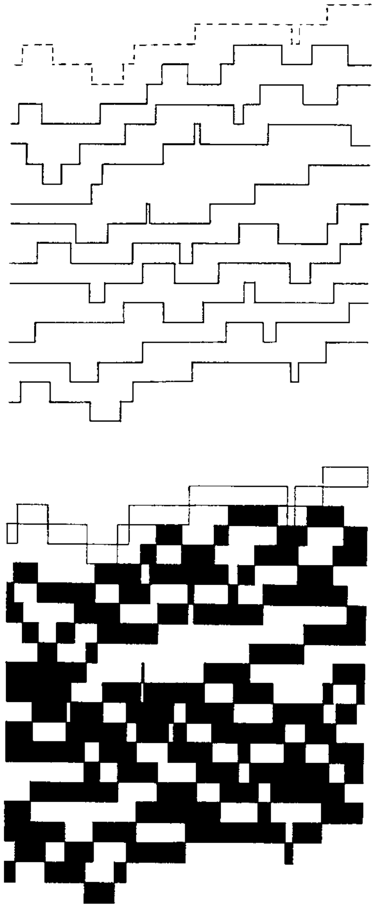

Fig. 1. (a) Contours representing a configuration of the lattice gas, and (b) the corresponding particle locations (black regions).

i.e., single occupancy of sites (SOS) is implied. In terms of contours, this creates a unit vertical spike on top of a contour, which represents 2 coincident, opposite kinks. The spikes arrive as a Poisson process in time, and are uniform along the allowed intervals of contours, with rate i spikes per unit distance per unit time.

In the second type of transition, upkinks move with speed g to the left and downkinks move with speed g to the right. These represent the vertical displacement of particles that lie next to kinks. They evidently never violate the SOS rule. Neighbouring kinks moving together and combining constitutes a further transition. Upkinks that reach the left boundary reappear on the same contour at the right boundary, and downkinks that reach the right boundary reappear on the same contour at the left boundary.

One can relate the model to a discrete particle lattice gas as follows. Let M denote the number of particle sites across the rectangle. Particles are strongly attracted in the horizontal direction but less so in the vertical direction. Consequently particles tend to form long horizontal rows of neighbours. Equivalently, transitions which break a row, occur at a low rate: ref. 6 suggests a rate of order L/M as $M \to \infty$. Transitions which merely shift a kink in a row by one horizontal step, occur at a high rate: ref. 6 suggests a rate of order M/L as $M \to \infty$. Then one views the model on a horizontal scale that is large compared to L/M but small compared to L, to obtain the continuum model. For a model with a single contour, one can make this rigorous.[6]

The model is equivalent to the crystal growth model described in refs. 5 and 8. There, the contours are constant height contours of a 3D surface.

3. STATIONARY STATES

Currently there does not exist an explicit statistical mechanics of the non-equilibrium steady states of particle systems in more than one dimension. The problem has always been an inability to find appropriate stationary distributions analogous to the Gibbs distributions. By contrast, we have an explicit stationary probability distribution of the new process,[5] viz.

$$\Pi(\mathbf{C}) = Z^{-1}\eta^{\mathcal{N}} \tag{3.1}$$

where $\mathcal{N} = \sum_{l=1}^{\Lambda} (2N_l + K)$ is the total number of kinks, $\eta = (i/2g)^{1/2}$, and $\mathbf{C}$ is confined to contours consistent with the SOS constraints and PBC's. The partition function

$$Z(\Lambda, \Omega, L, K, \eta) = \sum_{\mathbf{C}} \eta^{\mathcal{N}} \tag{3.2}$$

involves integrations over the x_{lr} and y_{lr} and sums over the N_l, ζ_l and $\mathcal{N}$.

Now that we have (3.1), an exact transport theory can be developed. The steady vertical current, or mean coloured area crossing a unit horizontal line in unit time, is evidently

$$J = \frac{g}{L\Omega} \langle \mathcal{N} \rangle = g\eta \frac{\partial \psi}{\partial \eta} \tag{3.3}$$

where $\langle\ \rangle$ denotes the mean with respect to $\Pi(\mathbf{C})$ and

$$\psi = \frac{1}{L\Omega} \log Z \tag{3.4}$$

It is natural to identify $-kT\psi$ as the *free energy density*. Pressure and chemical potential may then be defined like their equilibrium counterparts.

One is mainly interested in the large system limit $L, \Omega \to \infty$ with $\Lambda/\Omega \to \lambda$. The particle frequency in the horizontal direction does not appear explicitly in the model, but if one nominates μ particles per unit width of black strip, then the particle density is $\lambda\mu$. The current may then be calculated via a free fermion method. If K remains finite in the large system limit, its influence vanishes and[5, 7]

$$J \to J_0 \equiv (2ig)^{1/2} \pi^{-1} \sin(\pi\lambda) \tag{3.5}$$

If $i = 0$, and $K/L \to \kappa$, then no new kinks are created, leading to the obvious formula[5]

$$J \to g\kappa\lambda \tag{3.6}$$

For $K/L \to \kappa$ in general[8]

$$J \to [J_0^2 + (g\kappa\lambda)^2]^{1/2} \tag{3.7}$$

Correlation functions and other quantities can also be calculated.[8] The crystal "facets" described in ref. 9 now take the form of separate phases forming horizontal bands across the region.

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